WJEC (Eduqas) Physics A-level

Topic 2.8: Orbits and the Wider Universe

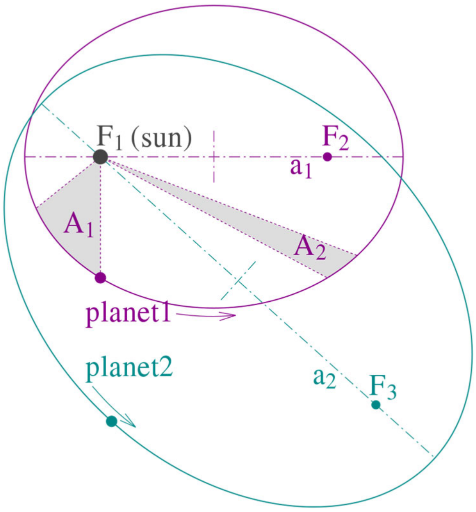
Notes

Kepler's Laws

Between the years of 1609 and 1619, Johannes Kepler published his laws of planetary motion that describe the motion of planets around the sun. These laws were based on **empirical data**, that is, they were not theoretically derived but collected by Tycho Brahe.

- 1st law - The orbit of each planet is an **ellipse** with the sun at one **focus**.
- 2nd law - The line joining the planet and the sun sweeps out **equal areas** in **equal time intervals**.
- 3rd law - The square of the **orbital period**, T, is proportional to the cube of the **semi-major axis**, a, of its orbit.

$$T^2 \propto a^3$$



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The figure displays the orbits of two different planets. The orbit of planet 1 (pink) is smaller than the orbit of planet 2.

- The Sun is at a focus, F_1, for both orbits in accordance with the 1st law.
- Areas A_1 and A_2 are equal in accordance with the 2nd law.
- The 3rd law tells us that the respective periods and semi-major axes of the orbits are such that:

$$\frac{{T_1}^2}{{T_2}^2} = \frac{{a_1}^3}{{a_2}^3}$$

where a_1 and a_2 are the semi-major axes of the orbits and have been marked on the figure.

Gravity and Kepler's Laws

We can use **Newton's law of gravitation to explain Kepler's laws** of planetary motion. It is assumed that the planets' motion is only influenced by the gravity of the Sun and the forces from other planets is negligible. The derivations of the first and second law are beyond the scope of A-level physics but we can derive the third law by approximating the elliptical orbit as a circular orbit. This allows us to deploy the equation for centripetal force.

Let M_{sun}, M_p, r, and v be the mass of the sun, the mass of a planet, the radius of the circular orbit and the velocity of the planet, respectively. The centripetal force that keeps the planet in orbit is the gravitational force that acts between the planet and the Sun. We can therefore equate the centripetal force and Newton's law of gravitation.

$$\frac{GM_{sun}M_p}{r^2} = \frac{M_p v^2}{r}$$

Here, G is the gravitational constant. Cancelling, we have

$$\frac{GM_{sun}}{r} = v^2$$

We know from circular motion that $v = \omega r$, where ω is the angular velocity of the planet. Substituting, we find

$$\frac{GM_{sun}}{r} = (\omega r)^2$$

$$\frac{GM_{sun}}{r} = \omega^2 r^2$$

$$GM_{sun} = \omega^2 r^3$$

Since $\omega = 2\pi/T$ where T is the period of the orbit,

$$GM_{sun} = \frac{(2\pi)^2}{T^2} r^3$$

$$T^2 = \frac{4\pi^2}{GM_{sun}} r^3$$

Each term in the fraction on the right-hand side is a constant. Thus,

$$T^2 \propto r^3$$

and we have proven Kepler's 3rd law using Newton's law of gravitation! For an elliptical orbit the orbital radius is replaced with the semi-major axis.

$$T^2 = \frac{4\pi^2}{GM_{sun}} a^3$$

$$T^2 \propto a^3$$

Using Orbits to Determine the Mass of Astronomical Bodies

In the previous section, we obtained the formula $T^2 = \frac{4\pi^2}{GM} r^3$. If we observe both the orbital period, T, and the orbital radius, r, we can calculate the mass of an astronomical body by rearranging our formula for M.

$$M = \frac{4\pi^2}{GT^2} r^3$$

For example, we can estimate the mass of the Earth by measuring the orbital period and radius of the Moon. It takes 27 days (2.3 million seconds) for the Moon to orbit the Earth once and the radius of orbit is approximately 385,000 km. Thus,

$$M_{Earth} = \frac{4\pi^2}{GT_{moon}{}^2} r_{moon}{}^3$$

$$M_{Earth} \approx 6.38 \times 10^{24} kg$$

which is in close agreement with the actual value! ($5.972 \times 10^{24} kg$).

We can also use Newton's law of gravitation to calculate the mass of an unknown orbiting object. For example, consider a satellite orbiting the Earth at a radius of 42,000km experiencing a centripetal force of $1.1 \times 10^3 N$. The mass of the satellite is calculated as

$$F = \frac{GM_{Earth}M_{sat}}{r^2}$$

$$M_{sat} = \frac{Fr^2}{GM_{Earth}}$$

$$M_{sat} = \frac{(1.1 \times 10^3)(42{,}000 \times 10^3)^2}{(6.67 \times 10^{-11})(6 \times 10^{24})}$$

$$= 4850kg$$

Rotation Curves of Galaxies and Dark Matter

Using Newton's law of gravitation and the equation for centripetal force, we can work out the velocity of an orbiting object in a spiral galaxy if we know mass of the galaxy, M, and the radius of the object's orbit, r.

$$\frac{GM \quad m}{r^2} = \frac{mv^2}{r}$$

$$\frac{GM}{r^2} = \frac{v^2}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

The **baryonic mass**, the mass made of baryons such as protons and neutrons, within a given radius, was measured for a number of galaxies. The expected velocity of material in the galaxy, obtained from the equation above, is displayed as the grey dashed line in the figure below.

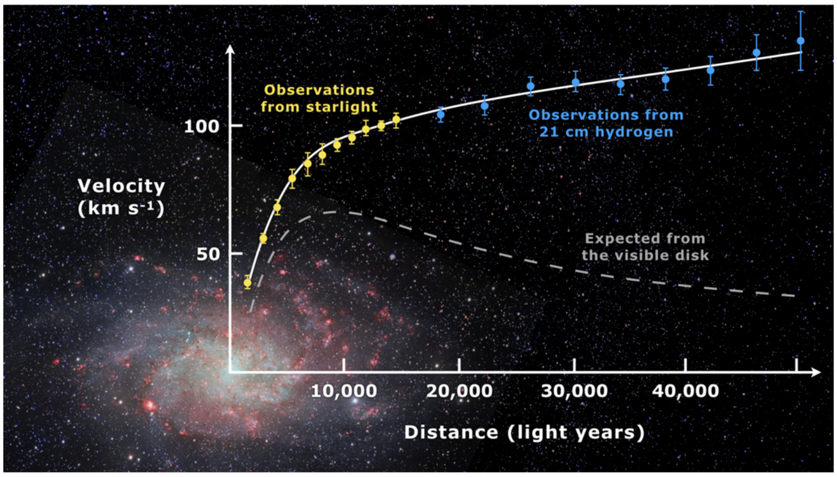

Image source: By Mario De Leo - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=74398525

The yellow and blue points in the figure represent experimental data. There is a clear disagreement with the theoretical grey line. The velocities are much larger than they should be - there doesn't seem to be enough mass within galaxies to hold them together.

At 40,000 light years the theoretical velocity is only a third of the measured velocity and, since our equation above tells us $M \propto v^2$, we are missing almost 90% of the galaxy's mass!

An explanation for this discrepancy is the presence of **non-baryonic dark matter** which is now thought to account for around 85% of all matter in the universe. This type of matter only interacts via the gravitational force making it difficult to detect - we usually detect matter by observing electromagnetic interactions.

Non-baryonic dark matter can be detected using **gravitational lensing,** the observation of distorted light from a distant source that has been bent by the mass of the galaxy as per Einstein's Theory of General Relativity. The amount of distortion is not consistent with the amount of mass that 'should' be in the galaxy, a further indication of a dark matter halo surrounding the galaxy.

It is theorised that dark matter would have to interact with the Higgs boson. Physicists are observing the decay of the **Higgs Boson** to see if there are any unexplained decay mechanisms that could suggest the presence of dark matter.

Mutual Orbits and Centre of Mass

When a body is orbiting another body we tend to picture the larger body stationary with the smaller body orbiting around it. A more accurate picture is both bodies orbiting around their **mutual centre of mass** (COM) which we can determine using the masses of the two bodies and their separation.
Consider two spherically symmetric bodies orbiting about their centre of mass.

1) The COM must lie on the line joining the centres of the two bodies.
2) The COM must lie between the two bodies as the centripetal force points towards it.
3) The angular velocity of each body must be equal. If one was to be rotating faster, both bodies could end up on the same side of the COM, violating condition 1).

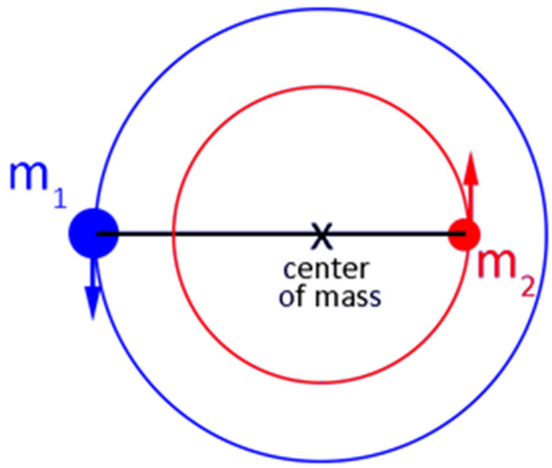

Image source: https://imagine.gsfc.nasa.gov/features/yba/CygX1_mass/binary/equation_derive.html

We can determine the COM using the following equation (on the formula booklet):

$$r_1 = \frac{M_2}{M_1 + M_2} d$$

where r_1 is the distance from the COM to object 1 with mass M_1 and d is the distance between the bodies. The corresponding equation for r_2 can be found by switching the M_1 and M_2 terms.

$$r_2 = \frac{M_1}{M_2 + M_1} d$$

For a circular obits we can calculate the mutual orbital period using another equation given in the equation booklet:

$$T = 2\pi \sqrt{\frac{d^3}{G(M_1 + M_2)}}$$

The Doppler Effect

An observer will observe a **change in frequency** and wavelength in waves being emitted from a source moving relative to the observer. This is known as the **Doppler effect**.
When the source is **moving towards** the observer, waves are observed to have an **increased frequency** and **decreased wavelength**.
When the source is **moving away** from the observer, waves are observed to have a **decreased frequency** and **increased wavelength**.
Astronomy is built upon the observation of **electromagnetic radiation.** For EM radiation the change in wavelength described by the Doppler effect is given by:

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$$

Where λ is the wavelength emitted by the source, $\Delta\lambda$ is the difference between λ and what the observer observes, v is the relative velocity of the source to the observer and c is the speed of light. **The above equation only works when $v << c$. As v approaches c, relativistic effects must be taken into account and the equation is no longer accurate.**

We can calculate the velocity of astronomical bodies relative to Earth by measuring the change in wavelength of EM radiation emitted by the body in accordance with the equation above.

Consider a star that we know is emitting light with a wavelength 430nm. A physicist on Earth measures a wavelength of 430.075nm. The velocity of the star is then:

$$v = \frac{c\Delta\lambda}{\lambda}$$

$$v = \frac{(3 \times 10^{8})(430.075 - 430) \times 10^{-9}}{430 \times 10^{-9}}$$

$$v = 5.2 \times 10^{4} ms^{-1}$$

If a star is moving away from Earth the Doppler shift is known as a **red shift**. If it is moving towards the Earth, the Doppler shift is known as a **blue shift**.

We can calculate the mass of orbiting bodies by observing the variation in their radial velocities. The velocity of the body will be different depending on if it is moving towards or away from Earth. We can combine the Doppler effect equation with the equations for the period and COM of the bodies to calculate the masses.

Hubble's Law

Astronomers have observed that all objects outside of our local cluster of galaxies appear red shifted, implying that all the objects are moving away from our galaxy.

Hubble's Law tells us that

$$v = H_0 D$$

where v is the radial velocity of an object relative to Earth, D is the distance from the object to Earth and H_0 **=** $67.8\ kms^{-1}Mpc^{-1}$ is the **Hubble constant**. Clearly, the radial velocity of an object is proportional to its distance from Earth.

We **can use H_0 to approximate the age of the universe!** Assuming the Big Bang theory is true, all positions in space started at the same point. The velocity and distance of a galaxy from Earth is then related to the time taken for the galaxy to travel this distance. Since $t = \frac{d}{v}$ we get that $\frac{1}{H_0}$ is approximately the time taken for galaxy travel this distance - approximately **13.8 billion years.**

Fate of the Universe and Critical Density

There are three possible outcomes for the universe that all depend on the **critical density, ρ_0.**

Firstly, we have an **open universe** that will continue to expand forever at a decreasing rate. This occurs if the gravitational force pulling the universe together is not strong enough to halt the expansion and will occur if the **density of the universe is less than ρ_0.**

Next, we have a **closed universe** in which the universe will eventually start to collapse into itself and return to the state it was in before the big bang - a process known as the **big crunch.** This occurs if the gravitational force is sufficient enough to decelerate the expansion and cause it to reverse. This will happen if the **density of the universe is greater than ρ_0.**

The third option is a **flat universe** which will occur if the gravitational forces are enough to halt the expansion, but not enough to reverse the process. This causes the universe to remain a constant size forever. For this to occur the **density of the universe must be equal to ρ_0.**

So what is ρ_0? We can **derive the critical density using conservation of energy**. Imagine a mass on the outskirts of the universe. If its kinetic energy is greater than its potential energy, it could 'escape' the universe suggesting an open universe. If its kinetic energy is less than its potential energy, it will eventually reverse direction and be unable to escape, suggesting a closed universe. For a flat universe, the kinetic energy would be equal to the potential energy:

$$\frac{1}{2}mv^2 = \frac{GMm}{D}$$

M is the mass of the universe, m is the mass of the object, D is the distance to the object, v is the velocity of the object and G is the gravitational constant. Substituting for v using Hubble's law,

$$\frac{1}{2}m(H_0D)^2 = \frac{GMm}{D}$$

$$\frac{1}{2}(H_0D)^2 = \frac{GM}{D}$$

Assuming a spherical universe, the mass is $M = \rho_0 V = \frac{4}{3}\pi D^3 \rho_0$ where ρ_0 is this critical density.

$$\frac{1}{2}(H_0D)^2 = \frac{G\frac{4}{3}\pi D^3\rho_0}{D}$$

$$\frac{1}{2}(H_0D)^2 = G\frac{4}{3}\pi D^2\rho_0$$

$$\frac{1}{2}{H_0}^2 = G\frac{4}{3}\pi\rho_0$$

$$\Rightarrow \rho_0 = \frac{3{H_0}^2}{8\pi G}$$